



Determination of Homogeneous Stiffness Matrix for Masonry Structure by means of Homogenizing Theorem

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Abstract

According to homogenizing theorem, heterogeneous or composite materials can be transformed to homogeneous materials by means of constant elastic classical displacements with effective elastic constants which indicate average characteristics of non-homogeneous environment. Most of the methods deliver the homogenized stiffness matrix as a function of constituent materials: mortar and brick, their volume fraction and the geometrical arrangement of masonry. In this paper, homogenized stiffness matrix for a single layer masonry wall is obtained by two existing homogenizing methods Reuss, & Voigt and an innovative proposed model. Then, the results are compared with the results of Lorenci Method & finite element analysis method. Finally, in the present study, the comparisons and analyses proved the accuracy and efficiency of innovative proposed model with detail.

Keywords: Homogenizing, Masonry, Modeling, Representative Volume Element, Finite Element

Introduction

Developing of a proper analytical method for masonry structure has two major consequences: a) in analyzing present masonry buildings and also masonry behavior of buildings with other structural systems; b) analysis of old and historical building's behaviour that maintenance and restoring of them is particularly important.

Most of the methods used in the masonry building analysis are based on finite element analyses. These methods have enough accuracy and give proper results but also are very time-consuming and need large memory space; hence makes them to be not applicable for real structures. As a result, in recent decades many investigations have been performed in order to conquer the issues related to building analysis. One of the appropriate solutions is homogenizing technique. By complete development of this method many analytical concerns may diminish and masonry analysis will be as plain as steel and concrete structure's analysis. In fact, the homogenizing technique is based upon mechanical properties and overall masonry behaviour according to mechanical and geometrical properties of constituent materials [1].

In this paper two of the existing methods and another proposed model are introduced. Then, the results are compared with each other, Lorenci method and finite element analysis. In the proposed model in addition to constituent's characteristics and their volume fraction, masonry geometrical arrangement is also considered. Moreover, elasticity coefficients ratio of brick to mortar have a dominant role in the case that mortar quality has decreased. The proposed model also considers this matter. It worth mentioning that many researchers have done lots of investigations on the context among them) [2] was founder of the method) [3-9] in linear field and [10] in nonlinear field, also for dynamic loading [11] can be referred. In the following sections, determination of the RVE is introduced and other explanations will follow after.

2. Representative Volume Element

RVE is the smallest unit in the masonry wall, that by its repetition in vertical and horizontal direction, a whole open wall will be made. In fact, the common procedure for developing the analysis of heterogeneous solids in micromechanics is to consider representative volume element which is a heterogeneous material volume, statistically represents the neighborhood of a certain point of continuum mass. In other words, a RVE will be put under homogenization theory after getting derived from a wall. Overall behaviour of the panel is analyzed according to the results from RVE. In this method masonry is considered as a composite, therefore the defined RVE should be small enough to represent structural microscopic details while being large enough to represent overall composite behaviour. As a result, the concept of RVE is applicable for estimation of continuum characteristics of a point of continuum mass. Although there is not a unique manner for RVE selection but two possible choices are shown in Fig. 1.

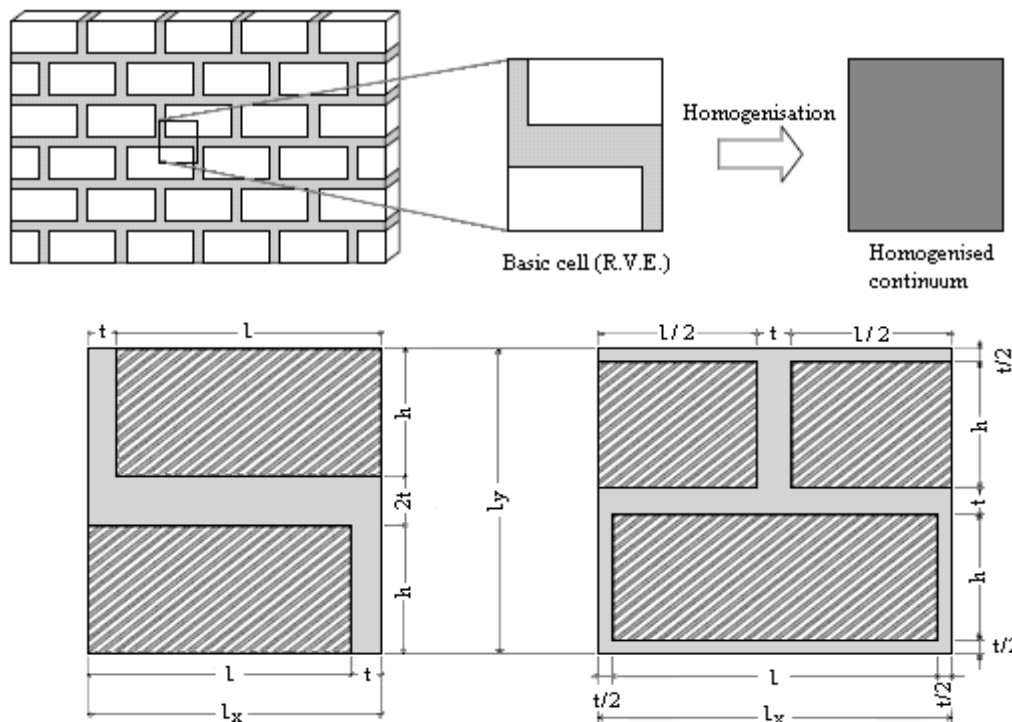


Figure 1: Basic cell selection for RVE [16]

It is assumed that, in terms of macro the RVE should be considered as a point in the wall; In addition, average stresses and strains in the element are regarded as stresses and strains of the point. However, according to the loading profile stresses and strains of different wall points may differ. Mechanical characteristics of masonry change consistent with the strain-stress curve of masonry. Consequently, numerical amounts of average stiffness matrix for different points of a wall through presented homogenization formulation will be different. With same reasoning, it could be deduced that as mechanical behaviour of mortar changes with the increase or decrease in imposed loading, for different stages of loading these amounts will be different. Therefore, by means of exact formulation of linear limits along with some non-linear considerations, the non-linear analysis of masonry wall could be achieved. The most vital mode in the procedure is to define precise linear model of RVE because each single small error occurred in the definition of average stiffness formulation can lead to large errors in the analysis of non-linear behaviour. This will impede further researches in much more simpler techniques for definition of linear RVE. Another important state of the procedure is to define an accurate collapse mechanism for homogenized masonry in order to develop all deformation modes of a masonry wall. Although in this technique, investigations for the first step have been acceptable but the second phase still need more focus and study. In this situation, experimental results can be helpful and grant much proper results where no other studies have reached.

Even with uniform traction boundaries or linear boundary displacements, displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x})$, strain field $\mathbf{E} = \mathbf{E}(\mathbf{x})$, and stress field $\mathbf{T} = \mathbf{T}(\mathbf{x})$ in RVE will change from point to point. It should be mentioned that in both situations of added surface data, RVE should be in equilibrium and its overall deformation compatible. The governing field equilibrium equation at a typical point \mathbf{x} in the volume V of the RVE is:

$$\nabla T(\mathbf{x}) = \mathbf{o}; \quad \mathbf{T}(\mathbf{x}) = \mathbf{T}^T(\mathbf{x}) \quad \text{in } V \quad (1)$$

Where body forces are absent and subscript T indicates transpose. Moreover, (∇) operator is defined as:

$$\nabla = \partial_i e_i = \frac{\partial}{\partial x_i} e_i \quad (2)$$

Besides equilibrium equations, strain-displacement equations should be considered as follows:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{2} \{ \nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T \} \quad \text{in } V \quad (3)$$

When self-equilibrium tractions are prescribed to the RVE boundary ∂V (as shown in Fig a-2), the following boundary equilibrium conditions should be confirmed:

$$\mathbf{T}(\mathbf{x})\mathbf{n} = \mathbf{t}^\circ \quad (4)$$

Where \mathbf{n} is the outer unit normal vector of the RVE boundary ∂V , and ∂V_t is the fraction of the RVE boundary where traction \mathbf{t}° is prescribed

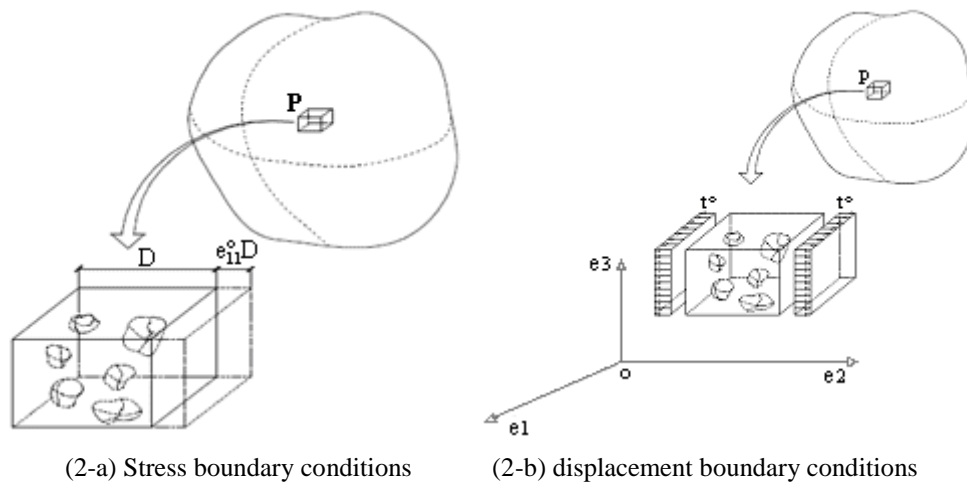


Figure 2: stress and displacement boundary conditions [17]

On the other hand, when it is assumed that self-consistent displacements (those displacements don't include rigid-body translations or rotations) are prescribed on the RVE boundaries, as it is shown in fig. b-2, it means:

$$\mathbf{u} = \mathbf{u}^\circ \quad \text{on} \quad \partial V_u \quad (5)$$

Where ∂V_u is the fraction of the RVE boundary where \mathbf{u}° is prescribed

After boundary-value problems associated with an RVE being formulated, the aim is to compute overall response parameters and to use them in order to define local characteristics of continuum material element, especially when it is necessary to start the problem in macro scale. In this scale, RVE represents one single point of continuum material where mechanical characteristics should be determined. However, as it is necessary to have uniform macro-fields on boundaries of RVE, the surface traction \mathbf{t}° have to be prescribed spatially uniform, or surface displacement \mathbf{u}° may be assumed spatially linear. In the first case, the aim is to find strain field as a function of corresponding prescribed nominal stress. Thus, components of the overall compliance tensor with the assumption of overall orthotropic behaviour of masonry would be as:

$$\begin{aligned} \bar{S}_{ijhk} &= \delta_{ij} \delta_{hk} \frac{\bar{\varepsilon}_{ij}}{\sigma_{hk}^{\circ}} && \text{Axial Components} \\ \bar{S}_{ijhk} &= \left(1 - \delta_{ij}\right) \left(1 - \delta_{hk}\right) \cdot \frac{\bar{\varepsilon}_{ij}}{\sigma_{hk}^{\circ}} && \text{Shear Components} \end{aligned} \quad (6)$$

Where consistent considerations are held for defining the proper average strain field with the intention to use in calculations. In the second case, the aim is to find the average stress field as a function of the corresponding prescribed nominal strain. As a result, the components of the overall stiffness tensor are:

$$\begin{aligned} \bar{C}_{ijhk} &= \delta_{ij} \delta_{hk} \frac{\bar{\sigma}_{ij}}{\varepsilon_{hk}^{\circ}} && \text{Axial Components} \\ \bar{C}_{ijhk} &= \left(1 - \delta_{ij}\right) \left(1 - \delta_{hk}\right) \cdot \frac{\bar{\sigma}_{ij}}{\varepsilon_{hk}^{\circ}} && \text{Shear Components} \end{aligned} \quad (7)$$

Where consistent considerations are done for defining the proper average stress field to use in the calculations.

3. Compliance tensors and overall elastic modulus

In this section, volume V of the Representative volume Element with boundary of ∂V is taken into account which consists of a uniform elastic mortar with elasticity and compliance tensors: C^M, S^M . Also it consists of n blocks with volume Ω^α where each block has elasticity and compliance tensors C^α, S^α ($\alpha = 1, 2, \dots, n$). It is supposed that blocks are perfectly jointed with mortar; all the constituents of the RVE are assumed linear. Hence, overall response of the RVE is linearly elastic. Overall compliance and elasticity tensors (average) of the RVE are indicated respectively by \bar{C}, \bar{S} and as follows are estimated in terms of RVE's micro-structural properties and geometry. As it was mentioned before, the prescribed macro-stress case and the prescribed macro-strain case are studied separately.

3.1. case of prescribed constant macro-stress

For the constant macro-stress ($\Sigma = \Sigma^\circ$) boundary tractions are:

$$t^\circ = n \cdot \Sigma^\circ \quad \text{on} \quad \partial V \quad (8)$$

Because of the heterogeneity characteristics of the masonry, neither of the stress fields nor strain fields produced in the RVE is uniform. Constant strain field E° is defined as:

$$E^\circ \equiv S^M : \Sigma^\circ \quad (9)$$

Considering that the actual stress field is denoted by $T(x)$ and the actual strain field is represented by $E(x)$, the following equations could be derived:

$$\begin{aligned} T(x) &= \Sigma^\circ + T^d(x) \\ E(x) &= E^\circ + E^d(x) \end{aligned} \quad (10)$$

Where $T^d(x)$ and $E^d(x)$ are variable stress and strain fields, represents disturbances or perturbations in the prescribed uniform stress field Σ° , which is due to heterogeneous RVE. Therefore, stress and strain tensors T and E are associated with each other by Hook's law:

$$T(x) = \Sigma^\circ + T^d(x) = \begin{cases} C^M : E(x) = C^M : \{E^\circ + E^d(x)\} & \text{in } M = V - \Omega \\ C^\alpha : E(x) = C^\alpha : \{E^\circ + E^d(x)\} & \text{in } \Omega^\alpha \end{cases} \quad (11)$$

$$E(x) = E^\circ + E^d(x) = \begin{cases} S^M : T(x) = S^M : \{\Sigma^\circ + T^d(x)\} & \text{in } M = V - \Omega \\ S^\alpha : T(x) = S^\alpha : \{\Sigma^\circ + T^d(x)\} & \text{in } \Omega^\epsilon \end{cases} \quad (12)$$

Where $\Omega = \bigcup_{\alpha=1}^n \Omega^\alpha$ is the overall volume of total blocks; M indicates mortar volume.

According to averaging theorem [8]:

$$\bar{T} = \langle T(x) \rangle = \Sigma^\circ \quad (13)$$

Where $\langle \rangle$ indicates average. From another point of view, overall average strain field is represented by:

$$\bar{E} = \langle E(x) \rangle = \langle E^\circ + E^d(x) \rangle \quad (14)$$

Hence, for the prescribed macro-stress:

$$\langle E^d(x) \rangle \neq 0 \quad (15)$$

In order to calculate the overall compliance tensor \bar{S} from:

$$\bar{E} = \bar{S} : \bar{T} = \bar{S} : \Sigma^\circ \quad (16)$$

The average amount of stress and strain field in each block and mortar should be calculated as follows:

$$\begin{aligned} \bar{E}^i &= \langle E(x) \rangle_i = \frac{1}{\Omega^i} \int_{\Omega^i} E(x) dV \\ \bar{T}^i &= \langle T(x) \rangle_i = \frac{1}{\Omega^i} \int_{\Omega^i} T(x) dV \end{aligned} \quad (17)$$

Where $i = \alpha, M$.

For the average stress and strain on mortar and blocks we have:

$$\begin{aligned} \bar{E}^M &= S^M : \bar{T}^M \\ \bar{E}^\alpha &= S^\alpha : \bar{T}^\alpha \end{aligned} \quad (18)$$

Therefore,

$$f_M \bar{E}^M = \bar{E} - \sum_{\alpha=1}^n f_\alpha \bar{E}^\alpha = \bar{S} : \Sigma^\circ - \sum_{\alpha=1}^n f_\alpha S^\alpha : \bar{T}^\alpha \quad (19)$$

And

$$f_M \bar{E}^M = f_M S^M : \bar{T}^M = S^M : \left\{ \Sigma^\circ - \sum_{\alpha=1}^n f_\alpha \bar{T}^\alpha \right\} \quad (20)$$

Consequently, it could be obtained that:

$$(S^M - \bar{S}) : \Sigma^\circ = \sum_{\alpha=1}^n f_\alpha (S^M - S^\alpha) : \bar{T}^\alpha = \sum_{\alpha=1}^n f_\alpha (S^M - S^\alpha) : \langle \Sigma^\circ + T^d(x) \rangle_\alpha \quad (21)$$

Where $f_\alpha = \frac{\Omega}{V}$ denote volume fraction of the α the block, and $f_M = \frac{M}{V}$ indicates volume fraction of the mortar.

Equation (21) leads us to an exact result in achieving \bar{S} . From (21), overall compliance tensor \bar{S} is defined in terms of average stress in blocks. The important point is that this result does not require entire stress field in each block, but only the estimation of average values in each block is necessary.

Since the overall response is linearly elastic, disturbances or perturbations in stress and strain fields, $E^d(x), T^d(x)$, are linear homogeneous functions of constant prescribed macro-stress Σ° . Therefore, we have:

$$(S^\alpha - S^M) : \langle \Sigma^\circ + T^d(x) \rangle_\alpha = (S^\alpha - S^M) : \bar{T}^\alpha = H^\alpha : \Sigma^\circ \quad (22)$$

Where constant fourth order tensor H is defined as:

$$(\bar{E}^\alpha - S^M) : \bar{T}^\alpha = \langle E^\circ + E^d(x) \rangle_\alpha - S^M : \langle \Sigma^\circ + T^d(x) \rangle_\alpha = H^\alpha : \Sigma^\circ \quad (23)$$

This is the change occurred in the average strain field Ω^α , if S^α is replaced by S^M . Since Σ° is arbitrary, by substitution of (22) in (21):

$$\bar{S} = S^M + \sum_{\alpha=1}^n f_\alpha H^\alpha \quad (24)$$

which is an exact result for a finite RVE as well as infinitely extended RVE. Restrictions of the above equation is homogeneous and elastic blocks and mortar, blocks should be without defects and being completely joint to mortar. However, some overall estimations and specializations are done in order to obtain constant tensors H^α ($\alpha = 1, 2, \dots, n$) [17].

3.2.case of prescribed constant macro-strain

For constant macro-strain $\varepsilon = \varepsilon^\circ$, boundary condition for the RVE is:

$$u^\circ = \varepsilon^\circ \quad \text{on } \partial V \quad (25)$$

Similarly, for this case the following equation is obtainable:

$$(C^M - \bar{C}) : \varepsilon^\circ = \sum_{\alpha=1}^n f_\alpha (C^M - C^\alpha) : \bar{E}^\alpha = \sum_{\alpha=1}^n f_\alpha (C^M - C^\alpha) : \langle \varepsilon^\circ + E^d(x) \rangle_\alpha \quad (26)$$

Equation (26) will lead to an exact result. This equation defines overall stiffness tensor \bar{C} according to average strains in blocks. The important point is that like the previous result, again having entire strain field in each block for acquiring \bar{C} is not required and the estimation average value in each block is needed.

As a result of linearly elastic overall response, disturbances and perturbations in stress and strain fields, $E^d(x), T^d(x)$, are linearly homogeneous functions of constant prescribed macro-strain ε° . Consequently, due to linearity if C^α is replaced by C^M the change in the average strain field in volume Ω^α is represented by:

$$\bar{E}^\alpha - S^M : \bar{T}^\alpha = J^\alpha : \varepsilon^\circ \quad (27)$$

Since ε° is arbitrary, following equation can be concluded from (26):

$$\bar{C} = C^M - \sum_{\alpha=1}^n f_\alpha C^M : J^\alpha = C^M : \left(1^4 - \sum_{\alpha=1}^n f_\alpha J^\alpha \right) \quad (28)$$

which is still an exact result. In this state, constant tensors such as H^α , J^α ($\alpha = 1, 2, \dots, n$) should be estimated.

4. Voigt and Reuss Estimating

In this section some overall conclusions are explained in micromechanical theorem framework. In particular, Hill (1952) has proved that the actual overall module independent from RVE geometry is located somewhere between Voigt and Reuss estimations [17, 18]. For simplicity, the simplest RVE with volume V is considered which consists of a linear elastic homogeneous mortar M , and a linear elastic homogeneous block Ω . Therefore, for both prescribed strain and stress boundary conditions, equation (21) will be as follows:

$$(S^M - \bar{S}) : \bar{T} = f_\Omega (S^M - S^\Omega) : \bar{T}^\Omega \quad (29)$$

where $F_\Omega = \frac{\Omega}{V}$ is a volumetric fraction of the block.

From (29), a unique dependence of the average values of the stress fields among blocks to the overall stress field in RVE volume can be obtained:

$$\bar{T}^\Omega = L^\Omega : \bar{T} \quad (30)$$

where:

$$L^\Omega = f_\Omega^{-1} (S^M - S^\Omega)^{-1} (S^M - \bar{S}) \quad (31)$$

Similarly, the average value of the stress field in mortar phase can be denoted as a function of overall stress field in RVE volume like:

$$\bar{T}^M = L^M : \bar{T} \quad (32)$$

where L^Ω and L^M indicates concentration matrices.

By taking into account equations (17) and (18), overall stress field in the RVE volume will be:

$$\bar{T}^M = f_M \bar{T}^M + f_\Omega \bar{T}^\Omega \quad (33)$$

From (33), by considering relations (30) and (32) we have:

$$f_M L^M - f_\Omega L^\Omega = I \quad (34)$$

where I is the unit matrix. As a result, concentration matrix L^M is expressed by:

$$L^M = (I - f_\Omega L^\Omega) f_M^{-1} \quad (35)$$

similarly, the simplest RVE with volume V consisting of a linear elastic homogeneous mortar M, with only one linear elastic homogeneous block Ω is taken into account. Again, for the prescribed stress and strain boundary conditions, equation (26) is assumed as follows

$$(C^M - \bar{C}) : \bar{E} = F_\Omega (C^M - C^\Omega) : \bar{E}^\Omega \quad (36)$$

From (36) a unique dependence of strain field values in block phase to the overall strain field in the RVE volume is obtained as:

$$\bar{E}^\Omega = M^\Omega : \bar{E} \quad (37)$$

$$M^\Omega = f_\Omega^{-1} (C^M - C^\Omega)^{-1} (C^M - \bar{C}) \quad (38)$$

In a similar manner the average value of the strain field in the mortar phase can be indicated as a function of RVE's volume as follows:

$$\bar{E}^M = M^M : \bar{E} \quad (39)$$

where M^Ω and M^M represents concentration matrices.

Considering (38) and (39) equations, overall strain field of the RVE's volume can be shown by:

$$f_M \bar{E}^M + f_\Omega \bar{E}^\Omega = I \quad (40)$$

by means of (40) and taking into account (37) and (38) relations:

$$f_M M^M + f_\Omega M^\Omega = I \quad (41)$$

Hence, concentration matrix M^M can be figured out as:

$$M^M = (I - f_\Omega M^\Omega) f_M^{-1} \quad (42)$$

According to Hook's law for average stress and strain in mortar and brick, also by means of (42), following equations can be derived from (33) and (40):

$$\begin{aligned} \bar{T} &= f_M C^M : \bar{E}^M + f_\Omega C^\Omega : \bar{E}^\Omega \\ \bar{T} &= f_M S^M : \bar{T}^M + f_\Omega S^\Omega : \bar{T}^\Omega \end{aligned} \quad (43)$$

Because:

$$\bar{T} = \bar{C} : \bar{E} \quad (44)$$

By equivalency of (44) with the first equation of (43) and according to (37) & (39), the effective RVE's stiffness tensor is obtained as:

$$\bar{C} = f_M C^M : M^M + f_\Omega C^\Omega : M^\Omega \quad (45)$$

Also because:

$$\bar{E} = \bar{S} : \bar{T} \quad (46)$$

By combining (46) with the second equation of (43) and by means of (30) & (32), the effective RVE's compliance tensor would be:

$$\bar{S} = f_M S^M : L^M + f_\Omega S^\Omega : L^\Omega \quad (47)$$

4.1. Voigt Estimation

A model for evaluating overall elastic stiffness tensor and most likely the simplest of them all is the one which was presented by Voigt on 1889 for the estimation of average constants of polycrystals:

$$\bar{E} = \bar{E}^M = \bar{E}^\Omega = E^\circ \quad (48)$$

Such conditions can be demonstrated by the simplified model in figure 3:

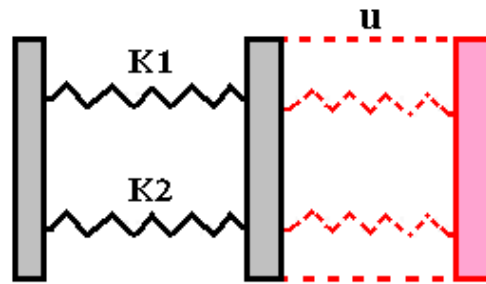


Figure 3: Prescribed strain problems

Considering (48) it can be deduced that:

$$M^M = M^\Omega = I \quad (49)$$

and (45) can be rewritten as:

$$\bar{C}^V = f_M C^M + f_\Omega C^\Omega \quad (50)$$

where superscript V on \bar{C} indicates that the (overall) effective stiffness tensor is obtained from Voigt estimation. Moreover, (27) with Voigt estimation would be as:

$$J^\Omega = (I^{(4)} - S^M : C^\Omega) \quad (51)$$

where substitution of equation (28) in (50) lead to the above mentioned equation.

4.2. Reuss Estimation

Correspondingly, a model for evaluating (overall) effective elastic compliance tensor and presumably the simplest of them all, is the one which was introduced by Reuss on 1929. He assumed that the overall stress field in the RVE is uniform (figure 4), which means:

$$\bar{T} = \bar{T}^M = \bar{T}^\Omega = T^\circ \quad (52)$$

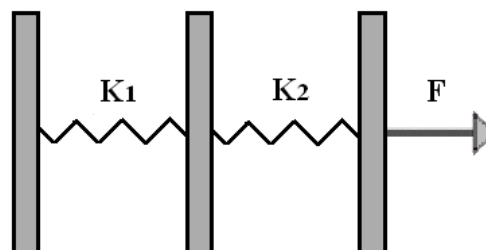


Figure 4: prescribed stress problems

Assuming equation (52), it will result in:

$$L^M = L^\Omega = I \quad (53)$$

Respectively, equation (47) will be in the form of (54):

$$\bar{S}^R = f_M S^M + f_\Omega S^\Omega \quad (54)$$

where superscript R states that the overall compliance tensor can be obtained from Reuss estimation.

4.3. Calculating numerical values for Voigt and Reuss estimations

Volume fractions can be defined as follows:

$$f_i = \frac{A_i}{A} \quad i = \text{mortar, brick} \quad (55)$$

where A_i is the surface of the mortar and brick in the xy plane, and A is the surface of the RVE in xy plane.

For the micromechanical model which is represented in the next section and according to the numerical data of Tables 1 and 2, we shall have:

$$\begin{aligned} f_m &= 0.2 \\ f_b &= 0.8 \end{aligned} \quad (56)$$

where f_m and f_b are volume fraction of mortar and brick, respectively for the RVE shown in figure 5 .

Regarding the overall compliance tensor derived from Reuss technique:

$$\bar{S}^R = f_m S^m + f_b S^b \quad (57)$$

where S^m and S^b are compliance tensors of isotropic homogeneous mortar and brick respectively. Reuss compliance tensor only shows three non-zero elastic coefficients that two of them are independent. Considering numerical data of (63), Reuss Compliance tensor would be:

$$S^R = 10^{-11} \begin{bmatrix} 14 & -2.1 & -2.1 & 0 & 0 & 0 \\ -2.1 & 14 & -2.1 & 0 & 0 & 0 \\ -2.1 & -2.1 & 14 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16.1 \end{bmatrix} \quad (58)$$

according to the above matrix, the Reuss stiffness tensor can be derived from the compliance tensor matrix inverse:

$$C^R = (S^R)^{-1} = 10^{11} \begin{bmatrix} 0.08 & 0.01 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0.08 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0.01 & 0.08 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.06 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.06 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.06 \end{bmatrix} \quad (59)$$

A similar procedure can be employed for the determination of the overall stiffness tensor with Voigt method:

$$\bar{C}^V = f_m C^m + f_b C^b \quad (60)$$

where C^m and C^b are mortar and brick stiffness tensor, respectively. In the same way, Voigt stiffness tensor shows three non-zero elastic coefficients that two of them are independent. Considering assumed numerical data of (63), the Voigt stiffness tensor can be calculated as:

$$C^V = 10^{11} \begin{bmatrix} 0.17 & 0.03 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.17 & 0.03 & 0 & 0 & 0 \\ 0.03 & 0.03 & 0.17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.14 \end{bmatrix} \quad (61)$$

Accordingly the compliance tensor can be obtained:

$$S^V = (C^V)^{-1} = 10^{-11} \begin{bmatrix} 6.21 & -0.93 & -0.93 & 0 & 0 & 0 \\ -0.93 & 6.21 & -0.93 & 0 & 0 & 0 \\ -0.93 & -0.93 & 6.21 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.14 \end{bmatrix} \quad (62)$$

The same matrix can be derived from finite element analysis results which have been performed with Ansys 11, then compare to the results of this section. In the upcoming section, the details of the model and the procedure of drawing stiffness matrix from analysis results will be explained.

5. Introducing micromechanical model and finite element analysis

According to figure 5, the micromechanical model which was used in the finite element analysis consists of a periodic RVE derived from one row masonry wall as shown in figure 1. Basic data in the analysis are summarized in Tables 1 and 2.

Table 1. Model dimensions

Phase	lx(m)	ly(m)	lz(m)
Brick	0.21	0.05	0.1
Model (RVE)	0.22	0.12	0.1
Mortar (t=0.01)	variable		

Table 2. Mechanical properties of the masonry

Phase	Material	Young's module E (N/m ²)	Poisson's coefficient, v	Eb/Em ^a
1	Brick	2x10 ¹⁰	0.15	10
2	Mortar	2x10 ⁹	0.15	

^a m indicates mortar and b stands for brick

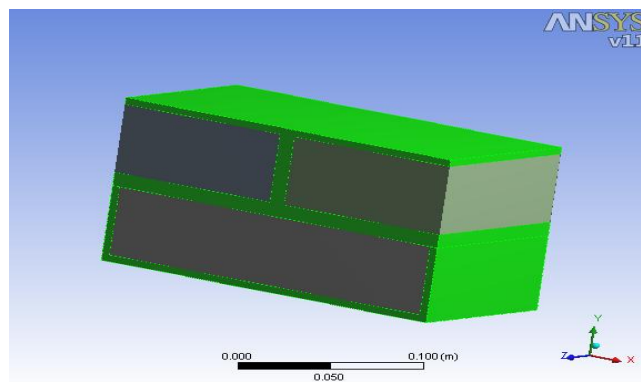


Figure 5: modelled RVE

Figure 6 demonstrated the meshing finite element (solid element) model used for numerical analysis. Moreover, it worth mentioning that these analyses are held in linear elastic limits along with considering bonded mortar and brick.

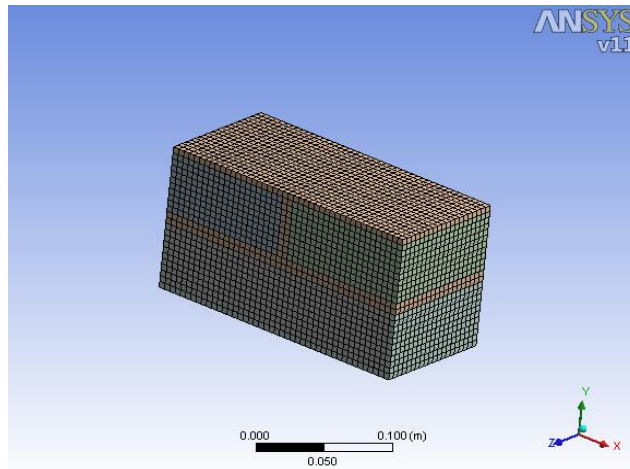


Figure 6: Meshing finite element model

6. Analysis with prescribed strain method

The aim of this analysis is to achieve overall stiffness tensor from 6 numerical analyses. Since, a mechanical orthotropic behaviour of the masonry structure is considered, therefore only 9 non-zero independent elastic coefficients must be determined.

It is presumed that:

$$\begin{array}{cccc}
 \varepsilon_1 \rightarrow \varepsilon_{xx} & \varepsilon_4 \rightarrow \varepsilon_{zy} & \sigma_1 \rightarrow \sigma_{xx} & \sigma_4 \rightarrow \sigma_{zy} \\
 \varepsilon_2 \rightarrow \varepsilon_{yy} & \varepsilon_5 \rightarrow \varepsilon_{zx} & \sigma_2 \rightarrow \sigma_{yy} & \sigma_5 \rightarrow \sigma_{zx} \\
 \varepsilon_3 \rightarrow \varepsilon_{zz} & \varepsilon_6 \rightarrow \varepsilon_{xy} & \sigma_3 \rightarrow \sigma_{zz} & \sigma_6 \rightarrow \sigma_{xy}
 \end{array} \quad (63)$$

In studies, the stress-strain relation can be formed as:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66} \end{bmatrix} \cdot \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix} \quad (64)$$

where superscript $[\bar{\quad}]$ means that the above equation refers to average values. By implementing six different loading conditions separately in six stages, the components of six columns of the average stiffness matrix are obtained according to the following equation:

$$\bar{C}_{ij} = \frac{\bar{\sigma}_i}{\varepsilon_j} \quad i, j = 1, 2, 3, 4, 5, 6 \quad (65)$$

While boundary conditions according is considered according to superficial displacement of the RVE, the

following formulas represents average strain in the RVE volume:

$$\bar{\varepsilon}_j = \frac{1}{V} \int_V \varepsilon_j \, dV = \varepsilon_j^\circ \quad (66)$$

Where V indicates volume of the RVE and ε_j° is the overall strain component as:

$$\varepsilon^{\circ} \cdot x = u^{\circ} \quad (67)$$

Where u is the prescribed superficial displacement. In order to obtain homogeneous stiffness tensor, the average strain in the RVE volume is calculated from:

$$\overline{\varepsilon}_j = \varepsilon_j^{\circ} = \sum_{r=1}^n \frac{\varepsilon_j^{(r)}}{n} \quad (68)$$

Where n is the number of overall RVE components that in this study it was 211120. In addition, $\varepsilon_j^{(r)}$ is the average of j th strain component for one sample element. Then, the average stress is calculated:

$$\overline{\sigma}_i = \frac{\sum_{r=1}^n \sigma_i^{(r)}}{n} \quad (69)$$

Where $\sigma_i^{(r)}$ is the average value of i th stress component of the sample element. Therefore, the characteristics of homogenized unit is obtainable from (65). As follows, the coefficients computed for the homogenized stiffness tensor for 6 different loading conditions is presented:

$$\begin{aligned} \overline{C}_{11} = \frac{\overline{\sigma}_1}{\varepsilon_1} = 0.13 \times 10^6 & \quad \overline{C}_{21} = \frac{\overline{\sigma}_2}{\varepsilon_1} = 0.013 \times 10^6 \\ \overline{C}_{31} = \frac{\overline{\sigma}_3}{\varepsilon_1} = 0.02 \times 10^6 & \quad \overline{C}_{41} = \overline{C}_{51} = \overline{C}_{61} = 0 \end{aligned} \quad (70)$$

Other components can be acquired similarly, therefore the homogeneous stiffness tensor would be as:

$$\overline{C}_{F.E.M} = 10^6 \begin{bmatrix} 0.13 & 0.013 & 0.02 & 0 & 0 & 0 \\ 0.013 & 0.08 & 0.014 & 0 & 0 & 0 \\ 0.02 & 0.014 & 0.18 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.06 \end{bmatrix} \quad (71)$$

Comparing C_{11} , C_{22} , C_{33} , it is observed that C_{33} has the most value because of the parallel action of mortar and brick in this situation. It means that when the stress or strain loading is prescribed in Z direction masonry stuff acts in parallel. Value of C_{11} is less than C_{33} which is for the reason that mortar and brick components act both in parallel and in series in this situation but for C_{22} as another surface exists that the force is transferred from the weak mortar component it is even lesser than C_{11} .

Comparing C_{44} , C_{55} , C_{66} it is perceived that C_{44} is approximately equal to C_{66} but the amount of C_{55} is considerably more than the other two. This is because of the parallel behaviour of mortar and brick under shear stress σ_{xy} which indicates that the average stiffness tensor component compared to brick behaviour (stiffness component) is influenced further. Additionally, for more analysis on outside the surface case, regarding the low amount of C_{22} and C_{66} ($C_{22} < C_{11}$, $C_{66} < C_{55}$), it can be observed that the likelihood of a walls' collapse because of its extensive height is more probable than the collapse due to the extended length. Besides, low value of C_{66} must not be disregarded which itself may cause a sliding collapse of the wall. As a matter of fact, in order to

illustrate a more complete and precise homogenized model, the significance of mortar role in determining some of the average stiffness matrix components should be taken into mind along with the volume combination of the two components and the elasticity coefficients fractions.

7. The proposed method

In order to obtain the homogenized stiffness matrix in homogenizing methods and in the proposed model, according to the presented formulas in the paper, the simulation mode of MATLAB software is employed.

The suggested method is based upon Voigt and Reusstechniques; in fact it is a combination of those methods amended by implementing coefficients to high elastic fractions of brick to mortar. Both Voigt and Reuss techniques are simple homogenizing methods that give out the homogeneous stiffness matrix for the masonry structure plane in the form of brick and mortar elastic stiffness. Both methods consider volume fractions of each in the derived RVE but the difference is that the Voigt method considers the parallel masonry while in Reuss technique masonry is presumed as series in all directions. However, regarding masonry arrangement it can be observed that masonry is not fully parallel or in series completely. Therefore, in order to obtain homogeneous stiffness matrix which depicts the best relation form of average volume stress in RVE with average strains, more accurate values can be acquired in this area. The difference is that the Voigt method considers the parallel masonry but in Reuss technique masonry is presumed as series in all directions. However, regarding masonry arrangement it can be observed that masonry is not fully parallel or in series completely. Therefore, in order to obtain homogeneous stiffness matrix which depicts the best relation form of average volume stress in RVE with average strains, more accurate values can be acquired in this area.

In the proposed model, some coefficients are defined for each component of the stiffness matrix. Using these coefficients the contribution amount of each corresponding matrix component calculated from Voigt and Reuss approach is estimated. Meanwhile, a novel formulation combined from Voigt and Reuss methods is presented. More explanations about determination and implementation of coefficients will be mentioned subsequently.

Another point which has not been considered in Voigt and Reuss method was $n = (E_b / E_m)$ ratio. Numerical Analyses showed that this will have influence on overall RVE behaviour. Although in low values of n the effect would be much less, but with higher values may cause large errors in determination of average stiffness matrix. Therefore, the problem is issued by a coefficient as follow:

$$e(n) = \frac{1}{\sqrt{\log(n+9)}} \tag{72}$$

Which for $n=1$ the result is one and for $n=\infty$ (the case E_m tends to zero) the result is 0. In fact, $n=\infty$ indicates the very poor mortar quality or the case that mortar is deformed and the softening phenomena has happened which in both cases the structure stiffness is reduced enormously (figure 7).

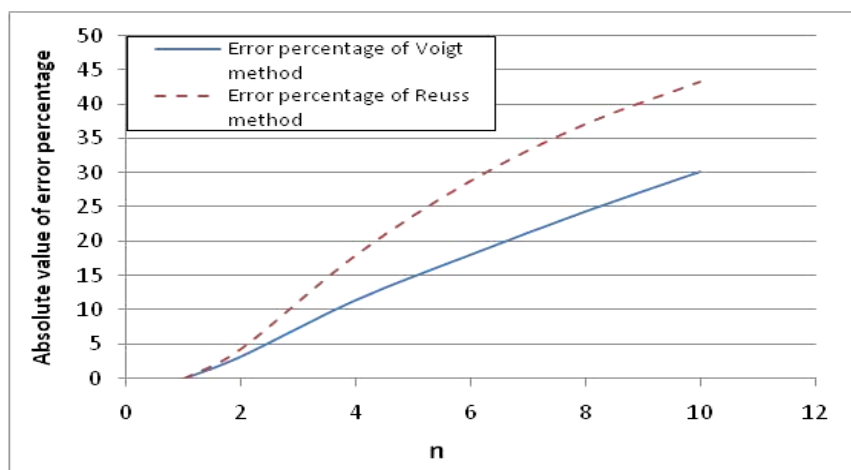


Figure 7: Increasing approximation in Voigt & Reuss method with increasing n value

In conclusion, the final equation for the average stiffness matrix of masonry plane in one point (RVE) is proposed in the following form. It will relate stresses and strains of that point (average stress and strain in the RVE is assumed as the point's stress and strain):

$$\bar{C}_{RV} = \frac{1}{\sqrt{\log(n+9)}} [R.\bar{C}^R + V.\bar{C}^V] \quad (73)$$

Where R and V are in the same format as \bar{C}^R and \bar{C}^V , and their corresponding components are multiplied to each other.

7.1. Determination of Effective Coefficients R and V

R matrix is assumed as:

$$R = \begin{bmatrix} R_x & (R_x + R_y) & (R_x + R_z) & 0 & 0 & 0 \\ & R_y & (R_z + R_y) & 0 & 0 & 0 \\ & & R_z & 0 & 0 & 0 \\ & sym & & R_{zy} & 0 & 0 \\ & & & & R_{xz} & 0 \\ & & & & & R_{xy} \end{bmatrix} \quad (74)$$

Components of the matrix are defined as follows:

R_i : Ratio of total volume of mortar joints within the RVE that their largest surface is perpendicular to displacement in i direction, to overall mortar volume (in other words ratio of total common surfaces perpendicular to displacement in i direction to overall common surface between brick and mortar).

R_{ij} : Ratio of total volumes of mortar joints in the RVE that their largest surface is perpendicular to one of i,j directions, to overall mortar volume (ratio of total common surfaces perpendicular to one of i,j directions, to total common surfaces between brick and mortar).

V matrix has the same form as R matrix that its corresponding non-zero components are calculated from:

$$V_{i,j} = 1 - R_{i,j} \quad (75)$$

Therefore, with R & V matrices \bar{C}_{RV} is obtainable from (73) for different values of n .

Because of less mortar diameter comparing to other dimensions, one surface of each mortar segments in the RVE has larger area than other surfaces and concurrently this is the common surface between brick and mortar. It is assumed that while the surface is perpendicular to displacement direction for determination of stiffness matrix, mortar and brick act in series and inversely while it is parallel to displacement direction, mortar and brick act in parallel too.

7.2. Numerical Results

The proposed formula for calculation of average stiffness matrix is represented as:

$$\bar{C}_{RV} = \frac{1}{\sqrt{\log(n+9)}} [R.\bar{C}^R + V.\bar{C}^V] \quad (76)$$

Where C^R and C^V are calculated for the RVE depicted in Figure 8. Now, R & V matrices should be found then is multiplied to the coefficients. The multiplying method is in a way that each $R_{i,j}$ components are multiplied to the corresponding components $C^R_{i,j}$, the same procedure is true for $V_{i,j}$ and $C^V_{i,j}$.

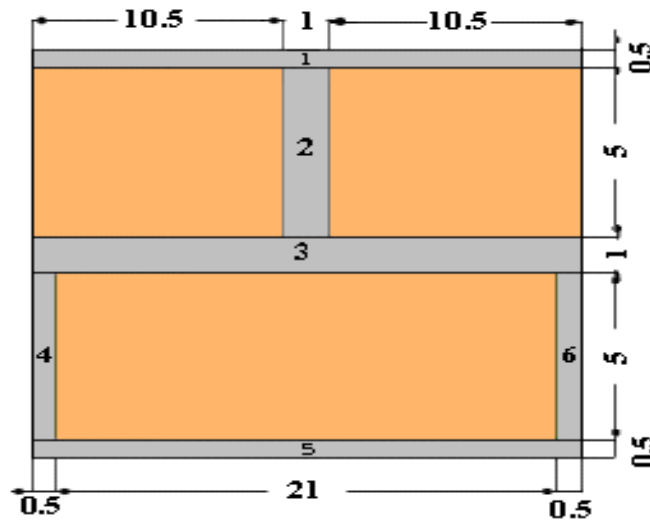


Figure 8: Numbering of mortar volumes in the RVE

The R matrix for the RVE shown in Figure 8 is calculated from:

$$\begin{aligned}
 R_x &= \frac{v_2 + v_4 + v_6}{v_m} = \frac{l_z \cdot \left[h \cdot t + 2h \frac{t}{2} \right]}{l_z \cdot [2h \cdot t + 2(l+t)t]} \\
 R_y &= \frac{v_1 + v_3 + v_5}{v_m} = \frac{l_z \cdot \left[(l+t)t + 2(l+t) \frac{t}{2} \right]}{l_z \cdot [2h \cdot t + 2(l+t)t]} \\
 R_z &= \frac{0}{v_m} = 0 \\
 R_{zy} &= \frac{v_1 + v_3 + v_5}{v_m} = \frac{l_z \cdot \left[(l+t)t + 2(l+t) \frac{t}{2} \right]}{l_z \cdot [2h \cdot t + 2(l+t)t]} \\
 R_{xz} &= \frac{v_2 + v_4 + v_6}{v_m} = \frac{l_z \cdot \left[h \cdot t + 2h \frac{t}{2} \right]}{l_z \cdot [2h \cdot t + 2(l+t)t]} \\
 R_{xy} &= \frac{v_2 + v_4 + v_6 + v_1 + v_3 + v_5}{v_m} = 1
 \end{aligned} \tag{77}$$

Where t is the mortar diameter, h brick height and l_z is the wall diameter in z direction. For the numerical data from Tables 1 and 2, R matrix according to (74) would be:

$$R = \begin{bmatrix} \frac{10}{54} & 1 & \frac{10}{54} & 0 & 0 & 0 \\ & \frac{44}{54} & \frac{44}{54} & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & sym & & \frac{44}{54} & 0 & 0 \\ & & & & \frac{10}{54} & 0 \\ & & & & & 1 \end{bmatrix} \quad (78)$$

And according to (75), the non-zero V matrix components would be:

$$V = \begin{bmatrix} \frac{44}{54} & 0 & \frac{44}{54} & 0 & 0 & 0 \\ & \frac{10}{54} & \frac{10}{54} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & sym & & \frac{10}{54} & 0 & 0 \\ & & & & \frac{44}{54} & 0 \\ & & & & & 0 \end{bmatrix} \quad (79)$$

By means of R & V matrices, also for n=10 the average stiffness matrix \bar{C}_{RV} is obtained from (76) as:

$$\bar{C}_{RV} = 10^{11} \begin{bmatrix} 0.137 & 0.0115 & 0.024 & 0 & 0 & 0 \\ 0.0115 & 0.083 & 0.014 & 0 & 0 & 0 \\ 0.024 & 0.014 & 0.153 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.068 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.113 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.055 \end{bmatrix} \quad (80)$$

8. Comparison and Analysis of the results

In table 3, a comparison between numerical results, obtained by finite element for the stiffness tensor components with results of other homogenizing methods mentioned in previous sections and another method based on micromechanical model in reference [16] (Lourenco method) is presented. According to the section 6, the reason of large errors in some tensor components in Voigt & Reuss homogenizing methods can be observed. In Fact, the most important thing that Reuss & Voigt methods have ignored is the arrangement order of masonry. Along with consisting materials characteristics and volume fraction of each, considering geometrical arrangement of masonry is essential for more accurate approximations. However, in Lourenco model three additional internal deformation modes concluded from finite element analysis are considered as well as geometrical arrangement while in the proposed model it has not been investigated. Although these additional deformation modes in the technique decrease the errors but considering them prevent from reaching to a closed formula which determines stiffness tensor components. On the other hand, computing numerical results for the mentioned method is possible by numerical techniques which need highly massive calculations and will be much more time consuming in nonlinear area. It worth mentioning that although the proposed model may not suffer from time consuming processes, but still lacks in other affairs. Not considering the additional deformation modes while the elasticity coefficient of brick to mortar is high will cause in larger errors because mortar lose its quality and the softening phenomena will happen. In Lourenco method which is the most exact of all, in order to obtain stiffness matrix, computations have a difficult procedure. On the contrary, the proposed model it is much easier to obtain the stiffness matrix. In the table below a comparison is made for all methods while n=10. It is obvious that the comparison is the same for compliance matrix.

Table 3. Comparison of results obtained for average stiffness tensor for a masonry by different methods ($\times 10^{11}$)

method	Finite element	Reuss		Voigt		Lourenco		Proposed method	
		Value	Error %	Value	Error %	Value	Error %	Value	Error %
c11*e11	0.133	0.0754	-43	0.1732	30	0.137	3.0	0.137	3
c21*e11	0.013	0.013	0	0.0306	135	0.013	-3.8	0.011	-12
c31*e11	0.023	0.013	-43	0.0306	33	0.023	0.9	0.024	5
c22*e11	0.076	0.0754	-1	0.1732	128	0.077	1.3	0.083	9
c32*e11	0.0146	0.013	-11	0.0306	110	0.015	1.4	0.014	-2
c33*e11	0.1658	0.0754	-55	0.1732	4	0.165	-0.5	0.153	-8
c44*e11	0.0681	0.062	-9	0.1426	109	0.067	-1.6	0.068	0
c55*e11	0.114	0.062	-46	0.1426	25	0.121	6.1	0.113	-1
c66*e11	0.064	0.062	-3	0.1426	123	0.064	0.6	0.055	-14

Conclusion

The homogenizing theorem for masonry structure is one of the techniques which have been introduced in recent decades. As a result, mechanical characteristics and overall masonry behaviour can be determined according to mechanical characteristics and geometry of constituents. The most important privilege of the model is fast structure analysis. This is due to utilization of homogenizing concepts and high speed engineering estimations. Moreover, the memory usage would decrease significantly.

In order to examine the mode, an RVE with the following characteristics was considered: a) RVE is a part of masonry wall by which repetition in two axis direction the whole wall takes shape; b) the optimum choice shall consists the most dominant features that have first-order effect on the overall sample characteristic and simultaneously hands in the most simple model.

In the present study, a RVE is presumed and a new method was proposed which was a combination of Voigt & Reuss method. Elasticity coefficients ratio of brick to mortar have a dominant role in the case that mortar quality has decreased. The proposed model also considers this matter. The model was verified with Voigt, Reuss, Lourenco, and finite element analysis.

Other homogenizing methods have excessive approximations in their process; Besides, where the concentrated stress exists and stress gradient is high the model encounters large errors.

In addition to characteristics of constituents and their volume fraction, considering masonry geometrical arrangement is essential in obtaining a more accurate estimation. In more accurate methods, internal additional deformation modes are also considered as well as geometrical arrangement of the masonry, which have been ignored in this study. Although considering additional deformation modes may decrease errors but prevents from acquiring a closed formula for determination of stiffness tensor components. Numerical results can be achieved by some numerical methods which due to massive computations in nonlinear area will cause time-consuming calculations.

However, although the proposed model may not have the time-consuming problem and the formulas can be manipulated in nonlinear computations (nonlinear analysis which does not need much iterations nor stiffness matrix amendment in order to reach convergence), but still suffers from large errors because of not considering additional deformation modes while the elasticity coefficient ratio of block to mortar is high and mortar lose its quality hence the softening phenomena takes place.

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