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Geometrically nonlinear free vibration of functionally graded beams

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Abstract
In this paper, the large-amplitude free vibration of clamped immovable thin beams made of functionally graded materials (FGM) is investigated using the energy method and a multimode approach. The mathematical formulation is based on Euler-Bernoulli beam theory and von Karman geometric nonlinearity. It is assumed that the material properties of the FGM beam vary through the thickness direction according to a simple power law. The equations of motion are derived by applying Lagrange's equations. Using the harmonic balance method (HBM), the equations of motion are converted into a non-linear algebraic form and are solved by an iterative numerical method. The numerical results obtained here concerning the fundamental mode shape linear and nonlinear resonant frequencies are presented and compared with those available in the literature. The numerical results show also that the fundamental nonlinear frequencies and the corresponding mode shapes are amplitude dependent and that the volume fraction index has a significant influence on the nonlinear vibration characteristics of the beam.

Keywords: Functionally graded materials (FGMs), Euler-Bernoulli beams, Geometrically nonlinear vibration, Harmonic balance method (HBM).

1. Introduction
Traditional composite materials have been used successfully in aircraft industry and other engineering applications because of their high strength to weight and stiffness to weight ratios. However, these composites are incapable to employ under the high-temperature environments. Recently, a new class of composites, known as functionally graded materials (FGMs), was developed. FGMs were initially designed as thermal barrier materials for aerospace structural applications and fusion reactors. They are now developed for general use as structural components in extremely high-temperature environments [1]. Functionally graded materials are inhomogeneous composites characterized by smooth and continuous variations in both compositional profile and material properties. The smooth variation of the mechanical properties along preferential directions is achieved by changing the volume fractions of two or more constituent materials, which are generally ceramic and metal(s). These materials do not contain well distinguished boundaries or interfaces between their different regions as in the case of conventional composite materials. Because of this, such materials posses good chances of reducing mechanical and thermal stress concentration in many structural elements, which can be developed for specific applications [2]. Practical methods used in graded materials technology to control the composition and microstructure have been discussed in Reference [2]. Recently, a review on fabrication techniques of functionally graded ceramic-metal materials has been made [3]. This study concluded that the powder metallurgy is actually the most suitable method for the manufacturing of FGMs.
In the last two decades, functionally graded materials have attracted considerable research efforts due to their increasing applications in many engineering sectors such as the design of aircraft and space vehicle structures amongst others. It is therefore important to investigate the static and dynamic behavior of FGM like structures such as beams, plates and shells which are used extensively as structural members. In the case of beams made of

FGMs (FG beams), many studies on the linear vibration behavior have been conducted [4-10]. However, since FG beams like structures are often subjected to severe dynamic loading, it is necessary to study their dynamic behavior at large amplitudes.

In the last few years, the large amplitude vibration of FG beams has attracted increasing research efforts. Ke et al. [11] studied the nonlinear free vibration of FG Euler-Bernoulli beams with different end supports using Galerkin's procedure. The nonlinear equations were based on Von Karman geometric nonlinearity and the governing equations were solved using direct numerical integration method and Runge-Kutta method. Simsek [12] investigated the non-linear forced vibration of an FG Timoshenko beam with pinned–pinned supports due to a moving harmonic load. The non-linear equations of motion were solved using Newmark-β method in conjunction with the direct iteration method. Fallah and Aghdam [13] studied large amplitude free vibration and post-buckling of FG beams resting on nonlinear elastic foundation subjected to axial force. Using Euler-Bernoulli beam theory and von Karman's strain-displacement relationship, a closed form solution of nonlinear governing equation has been presented by using a one term Galerkin's procedure and He's variational method. Some new results for the nonlinear natural frequencies and buckling load of the FG beams such as the effect of vibration amplitude, elastic coefficients of foundation, axial force, and material inhomogenity were presented for simply supported and clamped-clamped boundary conditions. Shooshtari and Rafiee [14] studied the nonlinear forced vibration problem of clamped clamed FG beams and the effects of different parameters on the frequency response have been investigated. Fallah and Aghdam [15] followed the same procedure as in [13] to analyze thermo-mechanical buckling and nonlinear free vibration of FG Euler-Bernoulli beams resting on nonlinear elastic foundation. Zerkane et al. [16] studied the nonlinear free vibration of clamped-clamped FG Euler-Bernoulli beams with immovable ends resting on elastic nonlinear foundation by using a homogenization procedure. This procedure permitted to reduce the problem of geometrically nonlinear free vibrations of functionally graded beams to that of isotropic homogeneous beams with effective bending stiffness and axial stiffness parameters. Hamilton's principle has been applied and a multimode approach was derived to calculate the fundamental nonlinear frequency parameters. Lai et al. [17] derived accurate analytical solutions for large amplitude vibrations of thin FG beams within the framework of the Euler-Bernoulli beam theory and the von Karman type geometric nonlinearity. A generalized perturbation technique was employed to solve the formulated nonlinear equations. Based on Euler-Bernoulli beam theory and von Karman geometric nonlinearity, Taeprasartsit [18] used the finite element method to study the large amplitude free vibrations of thin FG beams. Applying a direct iterative method together with the principle of energy conservation, the generated eigenvalue problem was solved. Kanani et al. [19] investigated the large amplitude free and forced vibration of FG beam resting on nonlinear elastic foundation containing shearing layer and cubic nonlinearity. The theoretical formulations and governing partial differential equation of motion were derived based on Euler-Bernoulli beam theory and von Karman geometric nonlinearity. By assuming a uniformly distributed harmonic load, a single nonlinear ordinary differential equation with quadratic and cubic nonlinearities is obtained by using a one term Galerkin technique. Variational iteration method was used to derive closed form approximate solutions for both free and forced vibration. Very recently, an analysis on the large amplitude free vibration of axially functionally graded tapered slender beams under different boundary conditions was carried out in reference [20]. The dynamic behavior has been presented in the form of backbone curves in a dimensionless frequency-amplitude plane.

In the most of the above mentioned references, only single mode approaches were used. However, this assumption has been shown to be inaccurate [21-25], since the mode shape thus assumed is amplitude independent and therefore leads to linear patterns of the bending stress rather than the nonlinear patterns.

The present study focuses on the geometrically nonlinear free vibration of clamped-clamped FG beams using a multimodal approach. It is assumed that material properties follow a simple power law distribution through the thickness direction. Based on Euler-Bernoulli beam theory and von Karman's strain-displacement relationship, the mathematical formulation has been established using Lagrange's equations and the harmonic balance method. Consequently, the large-amplitude vibration problem is reduced to a set of nonlinear algebraic equations in terms of the contribution coefficients of the transverse basic functions only. This set represents a nonlinear eigenvalue problem, which reduces to the well-known linear eigenvalue problem derived from Rayleigh–Ritz analysis when the nonlinearity is omitted. The nonlinear eigenvalue problem needs to be solved iteratively. The nonlinear iterative procedure described in References. [24, 25], known as the linearized updated mode method, is used here for accurate determination of the nonlinear resonant frequencies and the deflection shapes for the fundamental clamped-clamped FG beam nonlinear mode shape, at various non-dimensional amplitudes. The effect of the volume fraction index of the power law distribution on the nonlinear vibration behavior of FGM beams is investigated.

2. The mathematical model formulation

Consider the large amplitude transverse free vibrations of a beam of length L in the x direction and of rectangular cross-section with width b in the y direction and thickness h in the z direction as shown in Figure 1. The beam is made of a functionally graded material composed of metal and ceramic and is clamped at its immovable ends.

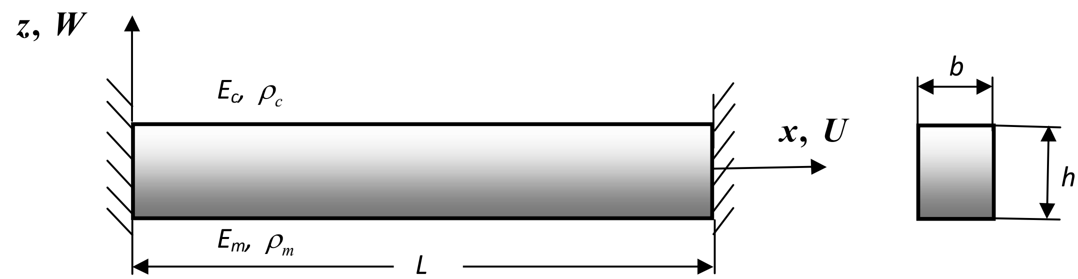

Figure 1: Geometry and notation of a functionally graded clamped-clamped beam

In this study a simple power law is considered to describe the variation of material properties from pure metal at the bottom face $z = -h/2$ to pure ceramic at the top face $z = +h/2$ of the beam as:

$$V_c = \left(\frac{2z+h}{2h}\right)^k = 1 - V_m \,, \quad (k \geq 0) \tag{1}$$

where V_c and V_m are the volume fraction of the ceramic and metal constituents respectively, and k is the volume fraction exponent which dictates the material variation profile through the thickness of the beam.
Therefore, the material properties of the beam, i.e., Young's modulus E and mass density ρ, vary continuously in the thickness direction (z-axis) as

$$E(z) = E_m + (E_c - E_m)\left(\frac{2z+h}{2h}\right)^k \tag{2}$$

$$\rho(z) = \rho_m + (\rho_c - \rho_m)\left(\frac{2z+h}{2h}\right)^k \tag{3}$$

where subscripts m and c refer to the metal and ceramic constituents respectively.
Based on the Euler–Bernoulli beam theory, the axial displacement, U, and the transverse displacement, W, of any point of the beam are given by

$$\begin{aligned} U(x,y,z) &= u(x,t) - z\frac{\partial w(x,\ t)}{\partial x} \\ W(x,y,z,t) &= w(x,t) \end{aligned} \tag{4}$$

where u and w are the axial and the transverse displacements of any point on the neutral axis of the beam, and t denotes time. When the amplitude of vibration is of the same order as the thickness of the beam, the linear beam theory must be extended to include the non-linear stretching force due to large displacement of the middle plane of the beam. In this case, the non-linear strain-displacement relation of the von Karman type is used

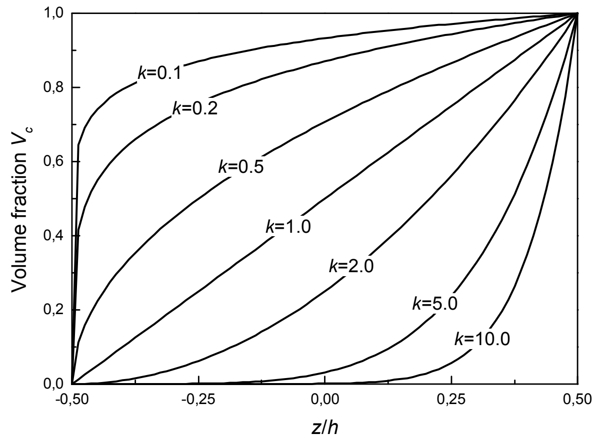

Figure 2: Variation of the volume fraction of the top ceramic surface, V_c, of the FG beam through the beam thickness in terms of the volume fraction index *k*.

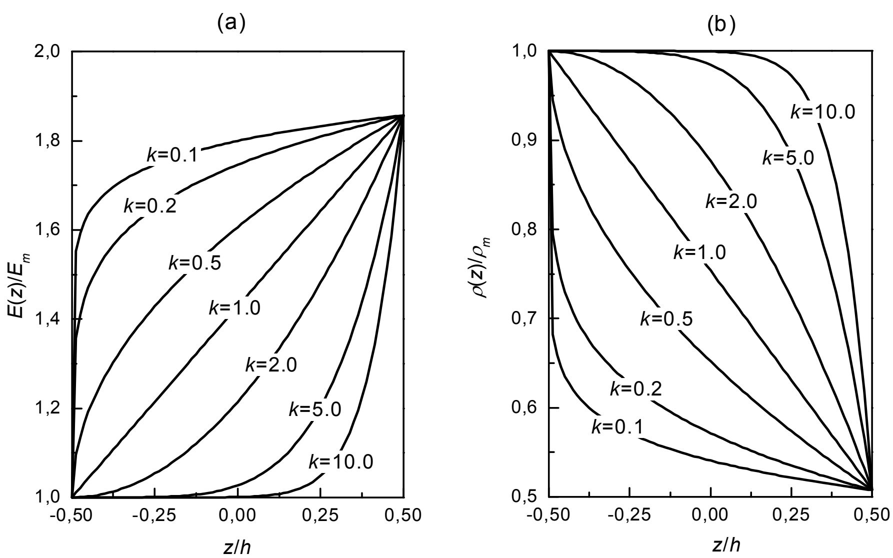

Figure 3: Variation of the material properties of the FG beam through the thickness for different values of the volume fraction index *k*: (a) Young's modulus; (b) Mass density.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \tag{5}$$

where $\varepsilon_{xx}^0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$ is the non-linear in-plane strain at the middle plane of the beam, and $\kappa_x = \frac{\partial^2 w}{\partial x^2}$ is the curvature of the beam.

The total strain energy V of the beam is given by

$$V = \frac{1}{2}\int_0^L (N\varepsilon_{xx}^0 + M\kappa_x)dx \tag{6}$$

where N and M are the normal force and bending moment resultants respectively, which are given by

$$N = bA_{11}\varepsilon_{xx}^{0} + bB_{11}\kappa_x \tag{7}$$

$$M = bB_{11}\varepsilon_{xx}^{0} + bD_{11}\kappa_x \tag{8}$$

where

$$(A_{11}, B_{11}, D_{11}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)(1, z, z^2)dz \tag{9}$$

The extensional stiffness (A_{11}), extensional-bending coupling stiffness (B_{11}) and bending stiffness (D_{11}) can be written in terms of the volume faction index k as follows

$$\begin{aligned} A_{11} &= h\left(E_m + \frac{1}{k+1}\left(E_c - E_m\right)\right) \\ B_{11} &= h^2\left(E_c - E_m\right)\frac{k}{2\left(k+1\right)\left(k+2\right)} \\ D_{11} &= h^3\left[\frac{E_m}{12} + \frac{k^2+k+2}{4\left(k+1\right)\left(k+2\right)\left(k+3\right)}\left(E_c - E_m\right)\right] \end{aligned} \tag{10}$$

Note that for a homogeneous isotropic beam, the extensional-bending coupling stiffness (B_{11}) is identically zero.
Inserting the expressions of N and M in equations (7) and (8) into equation (6), one obtains the following expression of the total strain energy V in terms of the mid-plane displacements u and w

$$V = \frac{1}{2}\int_0^L b\left\{A_{11}\left(u_{,x}^2 + \frac{1}{4}w_{,x}^4 + u_{,x}w_{,x}^2\right) - B_{11}\left(2u_{,x}w_{,xx} + w_{,xx}w_{,x}^2\right) + D_{11}\left(w_{,xx}\right)^2\right\}\mathrm{d}x \tag{11}$$

in which comma denotes partial derivative with respect to the variable indicated.
For slender beams, rotary inertia can be neglected and the kinetic energy of the FG beam can be written as

$$T = \frac{1}{2}\int_0^L bI_0\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x \tag{12}$$

where the inertial coefficient I_0 is given by

$$I_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)dz \tag{13}$$

which can be expressed in terms of the volume fraction index k and the mass density of metal and ceramic constituents of the FGM beam as

$$I_0 = h\left(\rho_m + \frac{1}{k+1}\left(\rho_c - \rho_m\right)\right) \tag{14}$$

Using a generalised parameterisation and the usual summation convention, one can put

$$\begin{aligned} u(x,t) &= q_i^u(t)u_i(x) \qquad i = 1 - p_u \\ w(x,t) &= q_i^w(t)w_i(x) \qquad i = 1 - p_w \end{aligned} \tag{15}$$

where u_i and w_i are the in-plane and the out-of-plane basic functions; q_i^u and q_i^w are the corresponding generalized parameters, respectively; p_u and p_w are the number of in-plane and out-of-plane basic functions used in the model. Insertion of equations (15) into equations (11) and (12) leads to the following discretized expressions of the total strain energy V and the kinetic energy T, respectively

$$V=\frac{1}{2}\left\{q_i^w q_j^w k_{ij}^1+q_i^w q_j^w q_k^w q_l^w b_{ijkl}^1+q_i^u q_j^u k_{ij}^2+q_i^w q_j^u k_{ij}^3+q_i^w q_j^w q_k^u c_{ijk}^1+q_i^w q_j^w q_k^w c_{ijk}^2\right\}$$
$$T=\frac{1}{2}\left(\dot{q}_i^w \dot{q}_j^w m_{ij}^1+\dot{q}_i^u \dot{q}_j^u m_{ij}^2\right) \quad (16)$$

In the above expressions m_{ij}^1, m_{ij}^2, k_{ij}^1 and k_{ij}^2 are, respectively, the general terms of the mass and the linear rigidity tensors associated with the transverse and the axial displacements, respectively. b_{ijkl}^1 is the general term of the fourth order non-linearity rigidity tensor associated with the transverse displacement; k_{ij}^3 is the general term of the rigidity tensor representing the coupling between the in- and the out-of-plane displacements. c_{ijk}^1 is the general term of the third order non-linearity rigidity tensor representing the coupling between the in- and the out-of-plane displacements; c_{ijk}^2 is the general term of a further third order non-linearity rigidity tensor which is associated with the transverse displacement. The general terms of these tensors are given by

$$m_{ij}^1=bI_0\int_0^L w_i(x)w_j(x)\,\mathrm{d}x,\quad m_{ij}^2=bI_0\int_0^L u_i(x)u_j(x)\,\mathrm{d}x$$
$$k_{ij}^1=bD_{11}\int_0^L w_{i,xx}(x)w_{j,xx}(x)\,\mathrm{d}x,\quad k_{ij}^2=bA_{11}\int_0^L u_{i,x}(x)u_{j,x}(x)\,\mathrm{d}x$$
$$k_{ij}^3=-2bB_{11}\int_0^L u_{i,x}(x)w_{j,xx}(x)\,\mathrm{d}x,\quad c_{ijk}^1=bA_{11}\int_0^L w_{i,x}(x)w_{j,x}(x)u_{k,x}(x)\,\mathrm{d}x \quad (17)$$
$$c_{ijk}^2=-bB_{11}\int_0^L w_{i,xx}(x)w_{j,x}(x)w_{k,x}(x)\,\mathrm{d}x,\quad b_{ijkl}^1=\frac{bA_{11}}{4}\int_0^L w_{i,x}(x)w_{j,x}(x)w_{k,x}(x)w_{l,x}(x)\,\mathrm{d}x$$

The equations of motion derived from Lagrange's equations can be written as follows

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(q_i^w)m_{ir}^1+q_i^w k_{ir}^1+\frac{1}{2}q_i^u k_{ir}^3+q_i^w q_k^u c_{irk}^1+\frac{3}{2}q_i^w q_j^w c_{ijr}^2+2q_i^w q_j^w q_k^w b_{ijkr}^1=0,\quad r=1\text{ to }p_w \quad (18)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(q_i^u)m_{is}^2+q_i^u k_{is}^2+\frac{1}{2}q_i^w k_{is}^3+\frac{1}{2}q_i^w q_j^w c_{ijs}^1=0,\quad s=1\text{ to }p_u \quad (19)$$

In these equations, the numbers r and s are varied from 1 to p_w and from 1 to p_u, respectively.
Non-dimensional formulation can be introduced by putting

$$w_i(x)=hw_i^*(x^*),\;\; u_i(x)=\beta hu_i^*(x^*),\;\; t=\left(\frac{L^2}{h}\sqrt{\frac{I_{00}}{A_{110}}}\right)\tau$$
$$(A_{11},B_{11},D_{11})=A_{110}(A_{11}^*,hB_{11}^*,h^2D_{11}^*) \quad (20)$$

where τ is the non-dimensional time parameter and $\beta=h/L$ is a non-dimensional geometrical parameter representing the ratio of the beam thickness to its length. A_{110} and I_{00} are taken as the values of A_{11} and I_0 of a homogeneous beam. Equations (18) and (19) can be written in non-dimensional form as

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q_i^w)m_{ir}^{1*} + q_i^w k_{ir}^{1*} + \frac{1}{2}q_i^u k_{ir}^{3*} + q_i^w q_k^u c_{irk}^{1*} + \frac{3}{2}q_i^w q_j^w c_{ijr}^{2*} + 2q_i^w q_j^w q_k^w b_{ijkr}^{1*} = 0, \quad r = 1 \text{ to } p_w \tag{21}$$

$$\beta^2 \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q_i^u)m_{is}^{2*} + q_i^u k_{is}^{2*} + \frac{1}{2}q_i^w k_{is}^{3*} + \frac{1}{2}q_i^w q_j^w c_{ijs}^{1*} = 0, \quad \mathrm{s} = 1 \text{ to } p_u \tag{22}$$

The $m_{ij}^{1*}, m_{ij}^{2*}, k_{ij}^{1*}, k_{ij}^{2*}, k_{ij}^{3*}, c_{ijk}^{1*}, c_{ijk}^{2*}$ and b_{ijkl}^{1*} terms are non-dimensional tensors related to the dimensional ones by the following equations

$$\begin{aligned} (m_{ij}^1, m_{ij}^2) &= \rho_m bLh^3 (m_{ij}^{1*}, \beta^2 m_{ij}^{2*}) \\ (k_{ij}^1, k_{ij}^2, k_{ij}^3, c_{ijk}^1, c_{ijk}^2, b_{ijkl}^1) &= E_m \frac{bh^5}{L^3}(k_{ij}^{1*}, k_{ij}^{2*}, k_{ij}^{3*}, c_{ijk}^{1*}, c_{ijk}^{2*}, b_{ijkl}^{1*}) \end{aligned} \tag{23}$$

When the axial inertia is neglected, assumption generally used in the literature, equation (22) can be solved for the q_i^u's in terms of the q_i^w's. Substituting the resulting expression into equation (21) leads to the following set of p_w coupled non-linear ordinary differential equations in terms of the out-of-plane generalized parameters q_i^w only:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q_i^w)m_{ir}^{1*} + q_i^w k_{ir}^{*} + \frac{1}{2}q_i^w q_j^w c_{ijr}^{*} + 2q_i^w q_j^w q_k^w b_{ijkr}^{*} = 0, \quad r = 1 \text{ to } p_w \tag{24}$$

where the new tensors k_{ij}^*, c_{ijk}^* and b_{ijkl}^* are, respectively, the linear rigidity tensor, the third order non-linearity rigidity tensor and the fourth order non-linearity rigidity tensor. These tensors take into account the coupling between the in- and the out-of-plane displacements. It is to be noted here that equation (24) contains a quadratic nonlinearity mainly due to the bending-stretching coupling. However, for clamped-clamped beams either isotropic, composite or functionally graded, the quadratic term vanishes as reported by ke et al. [11] and fallah et al. [13]. Therefore, equation (24) reduces to the following set of coupled Duffing's equations

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q_i^w)m_{ir}^{1*} + q_i^w k_{ir}^{*} + 2q_i^w q_j^w q_k^w b_{ijkr}^{*} = 0, \quad r = 1 \text{ to } p_w \tag{25}$$

An exact mathematical solution of such set of coupled Duffing's equation can be obtained only in the one-dimensional case corresponding to the single-mode approximation, in terms of elliptic functions. In the multidimensional case, an approximate solution may be obtained by the harmonic balance method. It is assumed here that harmonic motion exists for moderate finite vibration amplitudes. Therefore, the out-of-plane generalized parameters can be written as

$$q_i^w(\tau) = a_i \cos(\omega^* \tau) = a_i \cos(\omega t) \tag{26}$$

where ω and ω^* are the dimensional and the non-dimensional nonlinear frequency parameters, respectively, which are related by

$$\omega^* = \left(\frac{L^2}{h}\sqrt{\frac{\rho_m}{E_m}}\right)\omega \tag{27}$$

Substituting equation (26) into equation (25) and applying the harmonic balance method leads to

$$a_i k_{ir}^* + \frac{3}{2}a_i a_j a_k b_{ijkr}^* - \omega^{*2} a_i m_{ir}^{1*} = 0, \quad r = 1 \text{ to } p_w \tag{28}$$

The set of nonlinear algebraic equations (28) can be written in a matrix form as

$$\left(\left[K^*\right]+\left[K^*_{NL}\right]\right)\{A\}=\omega^{*2}\left[M^{1*}\right]\{A\} \tag{29}$$

where the $p_w \times p_w$ matrices $\left[K^*\right]$ and $\left[M^{1*}\right]$ are respectively the rigidity and mass matrices. The non-linear rigidity matrix $\left[K^*_{NL}\right]$ is also of order $p_w \times p_w$. Each term of the non-linear rigidity matrix $\left[K^*_{NL}\right]$ is a quadratic function of the column matrix of coefficients $\{A\}=\left[a_1\, a_2 \cdots a_{p_w}\right]^T$: $(K^*_{NL})_{ij}=(3/2)a_k a_l b^*_{ijkl}$. It can be seen that when the nonlinear term is neglected, the nonlinear eigenvalue problem (29) reduces to the classical eigenvalue problem

$$\left[K^*\right]\{A\}=\omega^{*2}\left[M^{1*}\right]\{A\} \tag{30}$$

which is the Rayleigh–Ritz formulation of the linear vibration problem. In the linear case, the eigenvalue equation (30) leads to a series of eigenvalues and corresponding eigenvectors. In the nonlinear case, the solution of equation (24) should lead to a set of amplitude-dependent eigenvectors, with their amplitude-dependent associated eigenvalues. To solve the nonlinear eigenvalue problem (29), incremental-iterative methods are generally used. The iterative method of solution adopted here is the so-called linearized updated mode method [24, 25]. This method consists of solving successive linear eigenvalue problems by starting from the linear eigenvalue problem (30) until the convergence of the value of the eigenvalue ω^{*2} is achieved, leading also to the normalized eigenvector $\{A\}$, corresponding to the mode considered and according to the specified amplitude of vibration considered. It is to be noted that the nonlinear stiffness matrix $\left[K^*_{NL}\right]$ is calculated, in each iteration, from the scaled eigenvector according to the specified amplitude of vibration obtained at the centre of the beam.

3. Results and discussion

3.1. *Numerical details*

The basic functions must satisfy the clamped-clamped immovable boundary conditions studied here, i.e., $u=w=w_{,x}=0$. Since the non-linear fundamental mode shape of FG beams is of main concern in the present study, the out-of-plane basic functions are chosen as the symmetric clamped-clamped thin beam linear mode shapes. These functions are given by

$$w_i(x^*)=\frac{\cosh(v_i x^*)-\cos(v_i x^*)}{\cosh(v_i)-\cos(v_i)}-\frac{\sinh(v_i x^*)-\sin(v_i x^*)}{\sinh(v_i)-\sin(v_i)} \tag{31}$$

where the constants v_i are the eigenvalue parameters for a clamped-clamped beam given by solving numerically the transcendental equation $\cosh(v_i)\cos(v_i)=1$. The functions w_i are normalized in such a manner that: $\int_0^1 w_i^*(x^*)w_j^*(x^*)\mathrm{d}x^*=\delta_{ij}$.

The in-plane basic functions, which are anti-symmetric, are chosen as sine functions of the axial variable x^* and are given by

$$u_i^*(x^*)=\sin(2\pi i x^*) \tag{32}$$

Convergence studies, which are not presented here for brevity, of the spectral expansion appearing in equation (15) have shown that accurate results concerning the nonlinear frequency parameter are obtained when using a multimodal model with the numbers of basic functions: $p_u=p_w=6$.

3.2. *Linear vibration analysis*

To validate the present model in the linear case, a comparison is made here between the obtained results with those available in the literature. In this section, the functionally graded material of the beam is that considered in reference [10]. The top surface of the FG beam is ceramic rich (Al_2O_3: E_c=390 GPa, $\rho_c = 3960$ Kg/m^3), and the bottom surface of the FG beam is metal rich (Steel: E_m=207.08 GPa, $\rho_m = 7800$ Kg/m^3). The variation of the volume fraction of the ceramic constituent (V_c) is illustrated in Figure 2 against the beam thickness for various values of the power law index *k*. Three special cases are observed: k =1 indicates a linear variation of the composition between the top and bottom surfaces of the beam, k = 0 represents the case when the beam is made of full ceramic material of the top surface and infinite *k* represents the case when the beam is made of full metallic material of the bottom surface. The variation of material properties, Young's modulus *E* and mass density ρ , of the FG beam along the thickness is illustrated in Figure 3.

In reference [10], the non-dimensional frequency parameter λ used is related to the one used in the present work (ω^* in equation (27)) by

$$\lambda = \omega L^2 \sqrt{\frac{\rho_m A}{E_m I}} = \sqrt{12}\,\omega^* \qquad (33)$$

where *A* is the cross-sectional area of the beam and *I* is the second moment of area of the beam cross-section, respectively. Since the FG beam is of rectangular cross-section, *A* and *I* are related by: $I / A = h^2 / 12$.

Table 1 shows the non-dimensional fundamental natural frequency λ of the FG Beam with the clamped boundary conditions for different values of the volume fraction index *k* along with the results of [10] which used the dynamic stiffness formulation in the study of linear vibrations of Euler-Bernoulli FG beams. In the same table, results from reference [26], in which the finite element method was used, are also presented. A good agreement is noticed especially between the current results and those of reference [10]. The numerical results from [26] overestimate the fundamental FG beam frequency parameter. The result for large values of the power index *k* $(k = \infty)$ which simulates the case of full metallic isotropic beam is in excellent agreement with the classical result of Euler-Bernoulli beam theory.

Table 1: The non-dimensional fundamental natural frequency of a clamped-clamped FG beam for different values of the volume fraction index *k* ($\lambda = \omega L^2 \sqrt{\frac{\rho_m A}{E_m I}} = \sqrt{12}\,\omega^*$)

	Volume fraction index *k*							
	0.1	0.2	0.5	1.0	2.0	5.0	10.0	∞
Present	39.8811	37.6993	33.5856	30.3339	27.8504	25.7673	24.6133	22.3733
Ref. [10]	39.884	37.759	33.838	30.777	28.337	26.038	24.723	--
Ref. [26]	41.4272	40.5425	38.6984	37.0516	35.5109	34.0064	33.2975	--

Figure 4 shows the effect of the variation of the volume fraction index *k* on the non-dimensional fundamental frequency ω^* of the FG beam. It is seen that a nonlinear diminishing of the frequency occurs when *k* increased from 0 to 4 and a linear reduction occurs when *k* is increased from 4 to 10.

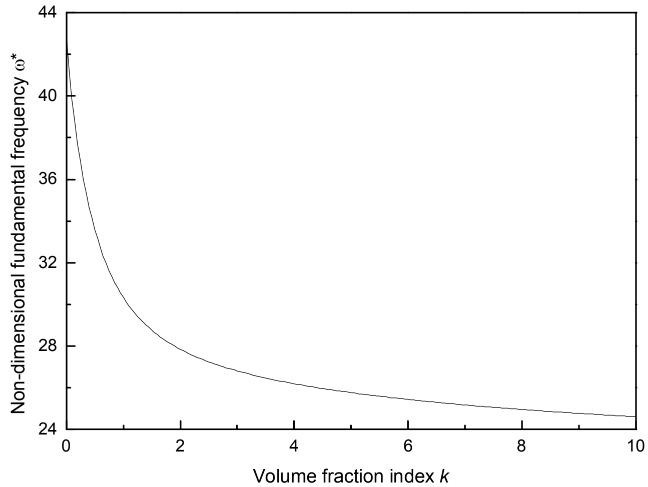

Figure 4: Variation of the non-dimensional frequency ω^* with the volume fraction index k for a clamped-clamped FG beam.

3.3. *Nonlinear vibration analysis*
In order to assess the validity of the present model in the nonlinear case, a first analysis is made for the geometrically nonlinear vibration of an isotropic clamped-clamped beam. Table 2 shows the fundamental nonlinear frequency ratios at various non-dimensional maximum amplitudes of vibration obtained at the centre of the beam. In this table, $R = \sqrt{I/A} = h/\sqrt{12}$ is the radius of gyration of the rectangular cross-section of the beam. It can be seen that the present results compare very well with those of Azrar et al. [27] and are close to those of the two term Galerkin's solution obtained in reference [11]. However, the results of Fallah et al. [13] overestimate the nonlinear to linear frequency ratios.

Table 2: Fundamental frequency ratios $\omega_{nl}^* / \omega_l^*$ of free vibration of an isotropic clamped-clamped beam at various vibration amplitudes

$w_{\max} / R$	Present	Azrar et al. [27]	Ke et al. [11]	Fallah et al. [13]
1	1.0222	1.0221	1.0232	1.0552
2	1.0858	1.0857	1.0894	1.2056
3	1.1833	1.1831	1.1907	1.4214
4	1.3064	1.3064	1.3185	1.6776
5	1.4478	1.4488	1.4656	--

In the case of geometrically nonlinear vibrations of clamped FGM beams, the functionally graded material of the beam is that considered in reference [11]. The top surface of the FG beam is ceramic rich (Silicon nitride Si_3N_4: E_c=322.2 GPa, $\rho_c = 2370$ Kg/m^3), and the bottom surface of the FG beam is metal rich (Stainless steel SuS304: E_m=207.7 GPa, $\rho_m = 8166$ Kg/m^3). The accuracy of the present model is checked by comparing the current nonlinear to linear frequency ratios $\omega_{nl}^* / \omega_l^*$ of the FG beam with previous published results. The results of reference [11] in the case of clamped-clamped FG beams were obtained by solving the following one dimensional Duffing's equation ($\gamma_b = 0$) in terms of elliptic functions

$$\ddot{w} + \gamma_a w + \gamma_b w^2 + \gamma_c w^3 = 0 \quad (34)$$

where γ_a, γ_b and γ_c are constants which were computed in reference [11] for some values of the volume fraction index *k* as shown in Table 3.

Table 3: Dimensionless coefficients $\gamma_a(\times 10^{-2})$, $\gamma_b(\times 10^{-2})$ and $\gamma_c(\times 10^{-2})$ of equation (34)

Volume fraction index *k*	γ_a	γ_b	γ_c
0.3	4.3057	0.0	3.0591
1.0	6.0727	0.0	4.4265
3.0	8.8965	0.0	6.6912

As mentioned before, the quadratic nonlinear term vanishes in the case of clamped-clamped beams ($\gamma_b = 0$). An approximate solution of equation (34) can be obtained by assuming a simple harmonic motion and applying the harmonic balance method. In this case, the nonlinear to linear frequency ratios as a function of the non-dimensional central amplitude of vibration of the beam is given by

$$\frac{\omega_{nl}^*}{\omega_l^*} = \left(1 + \frac{1}{16}\frac{\gamma_c}{\gamma_a}\left(\frac{w_{\max}}{R}\right)^2\right)^{1/2} \quad (35)$$

Table 4 shows the present nonlinear to linear frequency ratios at various non-dimensional amplitudes of vibration along with those obtained in references [11, 13] for a clamped-clamped FG beam. It can be seen that the present results are in good agreement with those of reference [11] obtained by using equation (35). However, the results of reference [13] overestimate the nonlinear to linear frequency ratios as in the isotropic case.

Table 4: Fundamental frequency ratios $\omega_{nl}^* / \omega_l^*$ of free vibration of the FGM clamped-clamped beam at various vibration amplitudes

$w_{\max} / R$	*k*=0.3		*k*=1			*k*=2		*k*=3	
	Present	Ref. [11]	Present	Ref. [11]	Ref. [13]	Present	Ref. [13]	Present	Ref. [11]
1	1.0227	1.0220	1.0225	1.0225	1.0559	1.0219	1.0545	1.0215	1.0232
2	1.0875	1.0852	1.0871	1.0873	1.2081	1.0847	1.2033	1.0832	1.0900
3	1.1869	1.1831	1.1860	1.1874	1.4262	1.1812	1.4170	1.1780	1.1929
4	1.3121	1.3079	1.3106	1.3149	1.6848	1.3030	1.6709	1.2980	1.3237
5	1.4557	1.4526	1.4536	1.4625	1.9679	1.4430	1.9493	1.4360	1.4748

The present model, which can be considered to be more accurate because of its multimodal character, can be exploited now to present further numerical results such as the influence of the volume fraction index on the fundamental nonlinear resonant frequency and the associated nonlinear mode shapes.
The dependence of the non-dimensional non-linear frequency on the non-dimensional vibration amplitude is plotted in Figure 5 for various values of *k*, for the fundamental non-linear mode shape of a clamped-clamped FG beam showing a hardening spring effect. The influence of the fraction index on the nonlinear to linear frequency ratio is more noticeable for higher values of the non-dimensional amplitude. For a given maximum amplitude of vibration, the nonlinear frequency ratio decreases with increasing values of the volume fraction index *k*.

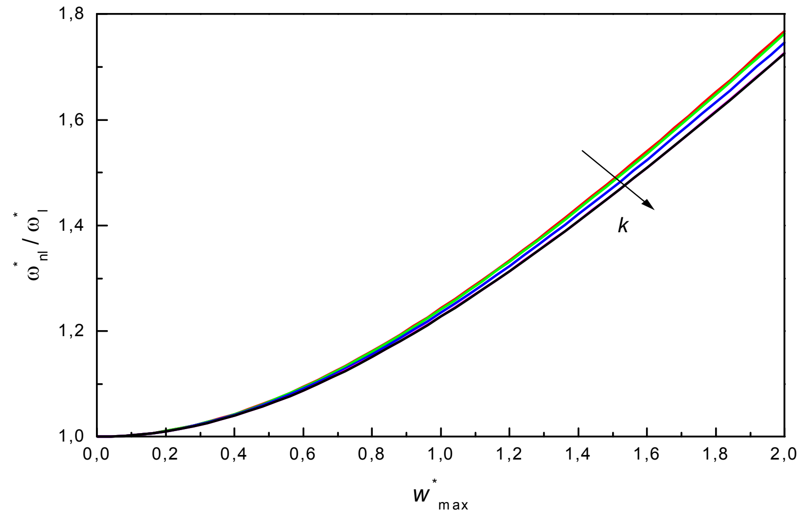

Figure 5: Backbone curves of the clamped-clamped functionally graded beam for different values of the volume fraction index *k*: 0.5, 1.0, 2.0, 5.0 and 10.0.

The fundamental nonlinear normalized mode shapes of a clamped-clamped immovable functionally graded beam for various values of *k*, and for various values of the non-dimensional amplitude of vibration ($w^*_{\max} = \frac{w_{\max}}{h}$) at the beam center, are plotted in Figures 6 and 7, respectively. In Figure 6, it can be seen that the influence of variation of *k* is negligible. However, as shown in Figure 7, the influence of the non-dimensional maximum amplitude of vibration on the mode shapes is clearly seen. The curves show also that the curvatures near the clamped edges increase by increasing the non-dimensional maximum amplitude. This result may lead one to expect that the bending stress near the edges of the functionally graded beam will increase nonlinearly with the increase of the vibration amplitude.

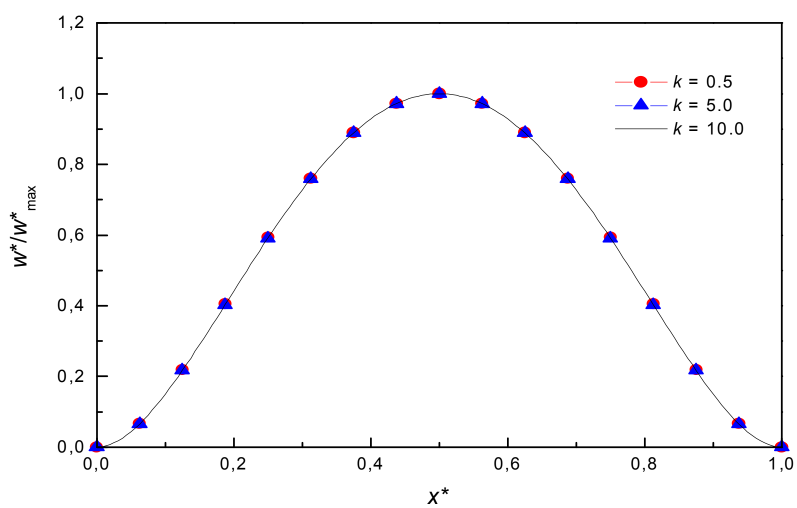

Figure 6: Fundamental nonlinear normalized mode shapes of the clamped-clamped functionally graded beam at a non-dimensional maximum amplitude $w^*_{\max} = 2.0$ and for different values of the volume fraction index k.

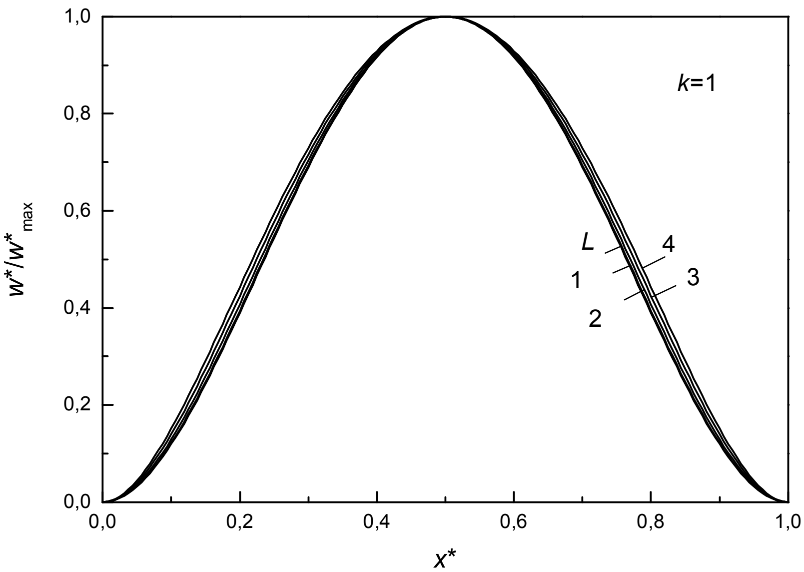

Figure 7: Fundamental nonlinear normalized mode shapes of the clamped-clamped functionally graded beam for different values of the non-dimensional maximum amplitude of vibration of the beam ($w^*_{\max} = \frac{w_{\max}}{h}$): *L*, small amplitude (linear); 1, $w^*_{\max} = 0.5$; 2, $w^*_{\max} = 1.0$; 3, $w^*_{\max} = 1.5$; 4, $w^*_{\max} = 2.0$.

Conclusions

The nonlinear free vibration problem of clamped-clamped functionally graded beams has been presented in this paper. The problem is studied within the framework of Euler-Bernoulli beam theory and the von Kármán type displacement-strain relationship. The materials properties are assumed to vary through the thickness direction of the FG beam according to a power law distribution. The governing equations have been derived by using Lagrange's equations and the harmonic balance method. When the in-plane inertia is neglected, the theory reduces the nonlinear free vibration problem of the FG beam to the solution of a set of nonlinear algebraic equations, in terms of the contribution coefficients of the transverse displacement only. This set of equations represents a nonlinear eigenvalue problem, which has been solved iteratively, for each specified amplitude of vibration, by the linearized updated mode method.

In the special case, when the nonlinear term is omitted, the model permitted the study of the free linear vibration problem of clamped-clamped FG beams. The results obtained for the fundamental natural frequency agree very well with the published results. It has also been shown that the volume fraction index has a significant influence on the fundamental natural frequency which decreases with increasing values of the volume fraction index.

In the nonlinear case, comparisons made between the present results and those from the published literature concerning the nonlinear to linear frequency ratios at various amplitudes of vibration, either in the isotropic and functionally graded cases, have shown that the nonlinear model presented here is very accurate. It has been shown that the fundamental nonlinear resonant frequencies and the associated mode shapes for the clamped-clamped FG beam are amplitude dependent. Particularly, the curvatures near the clamped edges increase by increasing the non-dimensional maximum amplitude. This result may lead one to expect that the bending stress near the edges of the functionally graded beam will increase nonlinearly with the increase of the vibration amplitude. It has also been shown that the volume fraction index has a negligible influence on the fundamental nonlinear mode shapes.

However, for a specified non-dimensional maximum amplitude of vibration, the fundamental nonlinear to linear frequency ratios decrease with increasing values of the volume fraction index.

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